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1985 J. Phys. A: Math. Gen. 18 2497

(http://iopscience.iop.org/0305-4470/18/13/025)

J. Phys. A: Math. Gen. 18 (1985) 2497-2508. Printed in Great Britain

On the application of the Born approximation to the Aharonov–Bohm and related problems

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Received 25 June 1984, in final form 6 March 1985

Abstract. The exact solution to the problem of scattering of electrons by a solenoid of infinite length is studied in two limiting cases; firstly, when the radius $r_0 \to 0$ with the enclosed flux ϕ held constant, and secondly, when $\phi \to 0$ with r_0 held constant. Previously reported discrepancies between the Born approximation and the Aharonov-Bohm solution of this problem are reconciled and conditions for the validity of each of these two approximations are established. The relevance of the results to a related scattering problem involving crystal dislocations is briefly discussed and in this case the Born approximation is shown to be valid over almost all the Fermi surface.

1. Introduction

The scattering of electrons by the magnetic (vector-potential) field of an infinitely long solenoid, whose circular cross section of radius r_0 contains magnetic flux ϕ, was investigated by Aharonov and Bohm (1959; henceforth AB) in the limit $r_0 \to 0$ with ϕ held constant. The parameter ϕ is a measure of the scattering strength and, at least for small ϕ, one expects the Born series for the scattering amplitude to converge rapidly. Yet Corinaldesi and Rafeli (1978; henceforth CR) have observed, that, when applied to the Hamiltonian considered by AB, the (first) Born approximation yields a scattering amplitude which differs† from that found by the latter authors. Although CR showed that the difference between the Born and AB approaches was confined to their different treatments of the s-wave scattering, their investigation did not reveal the underlying cause of the paradox, nor did it establish which, if either, of the two solutions was an appropriate representation of the true scattering amplitude.

It is the aim of the present paper to further investigate the cause of the above paradox and to ascertain the conditions under which either the Born or AB approximate solutions satisfactorily describe the true physical situation of electron scattering by a solenoidal magnetic field. The results contain a salutary reminder of the subtleties of limiting processes, especially those involving multiple limits—in this case the 'AB limit' ($r_0 \to 0$, $\phi =$ constant) and the 'Born limit' ($\phi \to 0$, $r_0 =$ constant), *both* of which are implied in the CR application of the Born approximation to the AB Hamiltonian.

† This discrepancy seems to have been first noticed by Feinberg (1963) and is also very briefly discussed by Ruijsenaars (1983). A similar discrepancy has been noted (Kawamura 1978a, Kawamura *et al* 1982) in connection with an analogous problem involving the interaction of conduction electrons with crystal dislocations.

0305-4470/85/132497+12\$02.25

We proceed by studying an exact solution of the magnetic scattering problem due to Kretzschmar (1965), whose treatment is outlined in § 2. In §§ 3 and 4 conditions for the validity of the AB and Born approximate solutions, respectively, are derived. Our conclusions are summarised, and their implications are discussed, in § 5. Some mathematical details are presented in an appendix.

2. Exact solution for arbitrary radius and flux

The treatment of this section is essentially that of Kretzschmar (1965), but for simplicity we set the core potential $V=0$. Some features of the more general case $V=\text{constant}\neq 0$ are discussed in § 5.

The Hamiltonian for a charge q in a magnetic vector potential $\boldsymbol{A}$ is $(\boldsymbol{p}-q\boldsymbol{A})^2/2m$, using SI units (Merzbacher 1970). We find that the two-dimensional time-independent wavefunction ψ satisfies, in cylindrical polar coordinates,

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}rq}{\hbar}A_\theta\right)^2+k^2\right]\psi=0 \tag{2.1}$$

where we have chosen the gauge div $\boldsymbol{A}=0$ and

$$\begin{aligned} A_r=0, \qquad A_\theta &= \phi/2\pi r, \qquad r\geqslant r_0 \\ &= \phi r/2\pi r_0^2, \qquad r\leqslant r_0 \end{aligned} \tag{2.2}$$

Upon seeking solutions of (2.1) in the form $\psi=R(r)\exp(\mathrm{i}n\theta)$ we find that for $r\geqslant r_0$ the function R satisfies Bessel's equation (Watson 1944) of order $(n-q\phi h^{-1})$. The general solution of (2.1) is therefore a linear combination of the functions

$$\psi_n^>(r,\theta)=\mathrm{e}^{\mathrm{i}n\theta}[a_nJ_{|n+\alpha|}(kr)+b_nY_{|n+\alpha|}(kr)], \qquad r\geqslant r_0 \tag{2.3}$$

for $n=0,\pm1,\pm2,\ldots$. Here a_n and b_n are arbitrary constants, J and Y denote Bessel functions of the first and second kinds, respectively, and

$$\alpha=-q\phi/h. \tag{2.4}$$

For $r\leqslant r_0$ the change of variables $z=\alpha r^2/r_0^2, f(z)=rR(r)$ reduces (2.1) to Whittaker's form of the confluent hypergeometric equation and the resulting solution for ψ which is finite at $r=0$ is a linear combination of the functions

$$\psi_n^<(r,\theta)=\mathrm{e}^{\mathrm{i}n\theta}r^{-1}M_{\kappa,\mu}(\alpha r^2/r_0^2), \qquad r\leqslant r_0. \tag{2.5}$$

Here

$$\kappa=(r_0^2k^2/4\alpha)-(n/2), \qquad \mu=|n/2| \tag{2.6}$$

and the Whittaker function is defined (Erdelyi *et al* 1953, § 6.9) as

$$M_{\kappa,\mu}(z)\equiv\mathrm{e}^{-z/2}z^{\mu+1/2}\Phi(\tfrac{1}{2}+\mu-\kappa,2\mu+1;z) \tag{2.7}$$

where

$$\Phi(a,c;z)=1+\frac{a}{c}z+\frac{a(a+1)}{c(c+1)}\frac{z^2}{2!}+\ldots \tag{2.8}$$

is Kummer's confluent hypergeometric series (Erdelyi *et al* 1953, § 6.1).

The continuity of ψ and $\nabla\psi$ at $r=r_0$ requires that

$$\frac{b_n}{a_n}=\frac{-A_n J_{|n+\alpha|}(kr_0)+J'_{|n+\alpha|}(kr_0)}{A_n Y_{|n+\alpha|}(kr_0)-Y'_{|n+\alpha|}(kr_0)} \tag{2.9}$$

where

$$A_n=k^{-1}\left(\frac{\partial}{\partial r}\ln[r^{-1}M_{\kappa,\mu}(\alpha r^2/r_0^2)]\right)_{r=r_0}=(kr_0)^{-1}[-1+2\alpha M'_{\kappa,\mu}(z)/M_{\kappa,\mu}(z)]_{z=\alpha} \tag{2.10}$$

which depends on n through (2.6). The primes in (2.9) and (2.10) denote derivatives with respect to the arguments kr_0 and z, respectively.

The physical significance of the exact solution (2.9) can be enhanced by expressing it in terms of phaseshifts. This will also enable us to establish criteria for the adequacy of the approximate solutions to be discussed in §§ 3 and 4. To establish reference phases we consider the solutions of (2.1) with $A_\theta\equiv 0$. Those solutions which are finite everywhere are

$$\psi_n^0(r,\theta)=\mathrm{e}^{in\theta}J_n(kr) \tag{2.11}$$

with asymptotic form (Watson 1944, § 7.21), as $kr\to\infty(kr\gg n)$

$$\psi_n^0\sim\mathrm{e}^{in\theta}(2/\pi kr)^{1/2}\cos[kr-(n+\tfrac{1}{2})(\pi/2)]. \tag{2.12}$$

On defining

$$\cos\Delta_n=a_n(a_n^2+b_n^2)^{-1/2},\qquad \sin\Delta_n=-b_n(a_n^2+b_n^2)^{-1/2} \tag{2.13}$$

the asymptotic form of (2.3) can be written as

$$\psi_n^>\sim\mathrm{e}^{in\theta}[2(a_n^2+b_n^2)/\pi kr]^{1/2}\cos[kr+\Delta_n-(|n+\alpha|+\tfrac{1}{2})\pi/2]. \tag{2.14}$$

The phase shift is thus

$$\delta_n=\Delta_n+(n-|n+\alpha|)(\pi/2). \tag{2.15}$$

3. The Aharanov–Bohm limit: $kr_0\to 0$

The solution adopted by AB, in the limiting case where $kr_0\to 0$ with α fixed, was $b_n=0$ for all n. Hence $\Delta_n=0$ also and the AB phaseshifts are, from (2.15)

$$\delta_n^{\mathrm{AB}}=(n-|n+\alpha|)\pi/2=\pm\pi\alpha/2(\text{modulo }\pi). \tag{3.1}$$

Comparing (3.1) with (2.15) we see that for the AB solution to closely approximate the exact solution requires

$$|\Delta_n|\equiv|\tan^{-1}(b_n/a_n)|\ll\pi|\alpha|/2 \tag{3.2}$$

for all n. Throughout the remainder of this paper it will be convenient to suppose† $|\alpha|\ll 1$ and in this case (3.2) reduces to

$$|b_n/a_n|\ll\pi|\alpha|/2. \tag{3.3}$$

† Extension to the case $\alpha\to$ integer $\neq 0$ is trivial.

Kretzschmar's (1965) analysis of (2.9) in the limit $kr_0 \to 0$ rests on his observation that as $kr_0 \to 0$ the square bracket on the right of (2.10) approaches a finite limit† so $|A_n| \to \infty$ as $(kr_0)^{-1}$. Then the series (Watson 1944)

$$J_\nu(x) = \sum_{m=0}^{\infty} (-1)^m (x/2)^{\nu+2m} [m!\Gamma(1+\nu+m)]^{-1} \tag{3.4}$$

shows that $J_\nu(x) \propto x^\nu$ as $x \to 0$ so that both terms in the numerator of (2.9) behave as $(kr_0)^{|n+\alpha|-1}$. Similarly, since (Watson 1944)

$$Y_\nu(x) = J_\nu(x)\cot(\pi\nu) - J_{-\nu}(x)\,\mathrm{cosec}(\pi\nu) \tag{3.5}$$

we see that $Y_\nu(x) \propto x^{-|\nu|}$ as $x \to 0$ so that both terms in the denominator of (2.9) behave as $(kr_0)^{-|n+\alpha|-1}$. Hence (2.9) is of order $(kr_0)^{2|n+\alpha|}$ and so

$$\lim_{kr_0 \to 0} b_n = 0 \tag{3.6}$$

for all integers n and $\alpha \neq$ integer. This justifies the AB solution ($b_n = 0$) and the corresponding boundary condition

$$\psi(0, \theta) = 0 \tag{3.7}$$

in the limit $kr_0 \to 0$, which may be regarded either as the long wavelength limit of the 'real' problem having a fixed r_0, or the mathematically convenient limit in which the radius of the solenoid is assumed to shrink indefinitely with the flux inside held constant.

Although the argument of the previous paragraph is sufficient to justify the AB solution in the limit $kr_0 \to 0$ it provides no quantitative estimate of just *how* small kr_0 must be for (3.3) to hold, i.e. for the AB solution to closely approximate the exact one. The following analysis provides such an estimate. It is convenient to assume henceforth that $kr_0 \ll 1$. This restriction and the aforementioned $|\alpha| \ll 1$ serve to simplify the analysis and are appropriate when comparing the AB and Born solutions.

Our analysis depends, as does Kretzchmar's argument which was sketched above, on replacing $Y_{|\nu|}(kr_0)$ and its derivative in the denominator of (2.9) by their leading terms from (3.5), namely,

$$Y_{|\nu|}(kr_0) \approx -\frac{\mathrm{cosec}(\pi|\nu|)}{\Gamma(1-|\nu|)}\left(\frac{kr_0}{2}\right)^{-|\nu|}, \qquad kr_0 \ll 1 \tag{3.8}$$

where we also used (3.4). It is clear from (3.5) that the approximation (3.8) is only permissible provided‡

$$|J_{-|\nu|}(kr_0)| \gg |\cos(\pi\nu) J_{|\nu|}(kr_0)| \tag{3.9}$$

and this is certainly not satisfied when $\nu \equiv (n+\alpha)$ approaches an integral value, for then $|J_{-|n|}| = |J_n|$. This clearly casts doubts on the adoption of the AB solution when $|\alpha|$ is sufficiently small—this will be investigated further in § 4. Continuing, we observe

† For $n \neq 0$ this limit depends only on α and n and Kretzschmar's argument is valid. For $n = 0$ the limit is $(-k^2 r_0^2/2)$ and leads to $A_0 \to -kr_0/2$ (see (3.15)) so $|A_0| \to 0$ as $kr_0 \to 0$. Nevertheless Kretzschmar's conclusions about $b_0 \to 0$ still apply.

‡ Because of the difficulty of imposing definite criteria as to what constitutes 'acceptable' accuracy we have used the imprecise comparison denoted by $\gg$. One should think of this symbol as denoting that the terms on either side of it differ by a factor of at least, say 10.

that when $kr_0 \ll 1$ the condition (3.9) becomes

$$\left(\frac{kr_0}{2}\right)^{-2|n+\alpha|} \gg \left|\cos[\pi(n+\alpha)]\frac{\Gamma(1-|n+\alpha|)}{\Gamma(1+|n+\alpha|)}\right| = \left|\frac{\pi(n+\alpha)\cot[\pi(n+\alpha)]}{[\Gamma(1+|n+\alpha|)]^2}\right| \tag{3.10}$$

where the final equality is a consequence of equation (1.2.6) of Erdelyi *et al* (1953). For small $|\alpha|$ we retain the leading term on the right of (3.10) to get the condition

$$\left(\frac{kr_0}{2}\right)^{-2|n+\alpha|} \gg \left|\frac{n+\alpha}{\alpha[\Gamma(1+|n|)]^2}\right|, \qquad kr_0, |\alpha| \ll 1. \tag{3.11}$$

This inequality is satisfied for all integers $n \neq 0$ if

$$(kr_0/2)^{2|n|} \ll |\alpha| \ll 1 \tag{3.12}$$

and for $n=0$ provided

$$(kr_0/2)^{2|\alpha|} \ll 1. \tag{3.13}$$

the latter being by far the more restrictive condition. Adopting it as a sufficient condition for the validity of (3.8), we get from (2.9) and (3.8)

$$\frac{b_n}{a_n} = -\frac{\Gamma(1-|n+\alpha|)}{\Gamma(1+|n+\alpha|)}\left(\frac{kr_0}{2}\right)^{2|n+\alpha|} \sin(\pi|n+\alpha|)\frac{|n+\alpha|-kr_0A_n}{|n+\alpha|+kr_0A_n}. \tag{3.14}$$

To further analyse (3.14) we need the kr_0 and α dependence of the A_n. It is shown in the appendix that, to first order in the small quantities α and $k^2r_0^2$,

$$kr_0A_n = |n| - \frac{k^2r_0^2}{2(1+|n|)} + \frac{\alpha n}{1+|n|} \tag{3.15}$$

and using this in (3.14) yields, to leading order in α and kr_0,

$$b_n/a_n = -(\pi\alpha n/2|n|)(kr_0/2)^{2|n|}[\Gamma(1+|n|)]^{-2}, \qquad n \neq 0 \tag{3.16}$$

and

$$b_0/a_0 = -\pi|a|(kr_0/2)^{2|\alpha|}. \tag{3.17}$$

Hence to obtain the AB regime (3.3) it is sufficient that (3.13) applies. That condition (3.13) is necessary as well as sufficient for the applicability of the AB solution is readily seen from the expression

$$-b_0/a_0 = \pi|\alpha|[1+(kr_0/2)^{-2|\alpha|}]^{-1}, \qquad kr_0, |\alpha| \ll 1 \tag{3.18}$$

which follows from (2.9) for kr_0 and $|\alpha| \ll 1$. We emphasise the severity of condition (3.13), which restricts kr_0 to extremely small values if the AB solution is to apply.

Finally we observe that (3.18) implies

$$b_0/a_0 = -\tfrac{1}{2}\pi|\alpha|(1-\varepsilon) \tag{3.19}$$

when $\varepsilon \equiv |\alpha| \ln(2/kr_0)$ is small compared to unity. This Born limit ($\alpha \to 0$) will be studied in § 4 using a different approach which is more closely related to the usual Born expansion in powers of α.

4. The Born limit: $\alpha \to 0$

As is well known, the Born approximation rests on the assumption that the change $\Delta\psi$, which is induced in the wavefunction by some perturbation, is small. In the present context this requirement becomes

$$\Delta\psi \to 0 \qquad \text{as} \qquad \alpha \to 0. \tag{4.1}$$

This is incompatible with (the s-wave component of) the AB boundary condition (3.7) which requires a large change, at $r=0$, in the incident wave of unit amplitude. It follows that the use of the AB limiting-case Hamiltonian in the familiar Born expression (e.g. Schiff 1968) for the (first-order) scattering amplitude is not justified. It is scarcely surprising that such a questionable procedure yields, as was pointed out by CR, results which disagree with the AB expression for the scattering amplitude.

To study the applicability of the Born approximation to the present problem we return to the exact solution (2.9) and consider the behaviour of the coefficients $b_n(r_0, \alpha)$ as $\alpha \to 0$. Since the conflict between (4.1) and (3.7) is due entirely to the s-wave($n=0$) component of the wavefunction (see also the remarks following (4.21)) it is sufficient to concentrate our attention on b_0. We observe also that the s-wave condition (3.13) is stronger than (3.12) in our earlier analysis. The importance of the s-wave contribution was previously emphasised by CR.

Since $J_\nu(x)$ is an analytic function (Watson 1944) of ν we can expand

$$J_{|\alpha|}(kr_0) = J_0(kr_0) + |\alpha|[\partial J_\nu(kr_0)/\partial\nu]_{\nu=0} + \ldots \tag{4.2}$$

and neglect higher-order terms provided $|\alpha|$ is sufficiently small. To see what 'sufficiently small' means we differentiate (3.4) m times (cf Watson 1944, p 61) to get, for $|x| \ll 1$,

$$[\partial^m J_\nu(x)/\partial\nu^m]_{\nu=0} = \left(\log\frac{x}{2}\right)^m + \mathrm{O}\left[\left(\log\frac{x}{2}\right)^{m-1}\right]. \tag{4.3}$$

Hence, the expansion (4.2) converges rapidly provided

$$|\alpha| \ln(2/kr_0) \ll 1 \qquad kr_0, |\alpha| \ll 1. \tag{4.4}$$

Conversely, if this is not the case the convergence is not rapid and higher-order terms in $|\alpha|$ must be retained. Condition (4.4) is thus seen to be both necessary and sufficient for the validity of the first Born approximation and, with kr_0 and $|\alpha| \ll 1$ it is almost always satisfied.

We henceforth assume that (4.4) holds and use the expansion (4.2), and the similar ones for Y, J' and Y', in (2.9) to find

$$\frac{b_0}{a_0} = -\frac{[(A_0J_0 - J_0')/(A_0Y_0 - Y_0')] + (\pi|\alpha|/2) + \mathrm{O}(\alpha^2)}{1 - [(\pi|\alpha|/2)(A_0J_0 - J_0')/(A_0Y_0 - Y_0')] + \mathrm{O}(\alpha^2)} \tag{4.5}$$

where the argument of all Bessel functions and their derivatives is kr_0. We used (Abramowitz and Stegun 1970, equation 9.1.68)

$$[\partial J_\nu(x)/\partial\nu]_{\nu=0} = \tfrac{1}{2}\pi Y_0(x), \qquad [\partial Y_\nu(x)/\partial\nu]_{\nu=0} = -\tfrac{1}{2}\pi J_0(x) \tag{4.6}$$

and the similar relations for J' and Y'. Since (see appendix)

$$(A_0J_0 - J_0')/(A_0Y_0 - Y_0') = \mathrm{O}(\alpha^2) \tag{4.7}$$

(4.5) yields (cf (3.19))

$$(b_0/a_0) = -(\pi|\alpha|/2) + O(\alpha^2), \qquad kr_0, |\alpha| \ll 1 \tag{4.8}$$

in the domain of kr_0 and α within which (4.4) is satisfied. Note that (4.8) differs from the AB solution, which is valid if and only if $|b_n/a_n| \ll \pi|\alpha|/2$. On using (4.8) in (2.13) we find $\Delta_0 = \pi|\alpha|/2$ so that the s-wave phaseshift (2.15) exhibits an analytic α dependence in the Born regime (4.4).

We now consider the application of the Born approximation to the Hamiltonian which corresponds to (2.1). This will have the dual purpose of allowing some discussion of the work of CR and of demonstrating that our small-α expansion which led to (4.8) is indeed equivalent to the Born approximation. The Hamiltonian corresponding to (2.1) is

$$H = -(\hbar^2\nabla^2/2m) + \Delta H \tag{4.9}$$

where

$$\frac{2m}{\hbar^2}\Delta H = \begin{cases} -r^{-2}[(2\mathrm{i}\alpha\, \partial/\partial\theta) - \alpha^2], & r \geqslant r_0 \\ -r_0^{-2}[(2\mathrm{i}\alpha\, \partial/\partial\theta) - (\alpha^2 r^2/r_0^2)], & r \leqslant r_0. \end{cases} \tag{4.10}$$

For comparison with the AB solution and that of CR we assume an incident wavefunction corresponding to a particle flux along the direction $\theta = \pi$. Then imposing the asymptotic condition (Aharanov and Bohm 1959)

$$\psi \sim \exp(-\mathrm{i}kr\cos\theta - \mathrm{i}\alpha\theta) + r^{-1/2}f(\theta)\exp(\mathrm{i}kr), \qquad kr \to \infty, \tag{4.11}$$

on the solutions of (2.1) leads in the usual way (e.g. Messiah 1961, § 19.5; the Green functions for the two-dimensional case are given by Economou 1979, § 1.2, and Morse and Feshbach 1953 § 7.2) to the exact expression

$$f(\theta') = -\frac{m\,\mathrm{e}^{\mathrm{i}\pi/4}}{\hbar^2(2\pi k)^{1/2}}\int_{-\pi}^{\pi} \mathrm{d}\theta \int_0^\infty r\,\mathrm{d}r \exp[-\mathrm{i}kr\cos(\theta - \theta']\Delta H\psi(r, \theta) \tag{4.12}$$

for the scattering amplitude.

The Born approximation uses the plane wave $\exp[\mathrm{i}kr\cos(\theta - \pi)]$ in the integrand of (4.12) in place of the exact solution of (2.1). On doing this, taking ΔH from (4.10) and using (McLachlan 1946, p 159)

$$\int_{-\pi}^{\pi} \mathrm{d}\theta \exp(-\mathrm{i}n\theta - \mathrm{i}z\cos\theta) = 2\pi\exp(-\mathrm{i}n\pi/2)J_n(z) \tag{4.13}$$

we readily find

$$f_{\mathrm{Born}}(\theta) = \mathrm{i}\alpha\pi[\exp(\mathrm{i}\pi/4)](2\pi k)^{-1/2}[I_1\tan(\theta/2) - \mathrm{i}\alpha I_0] \tag{4.14}$$

where

$$\begin{aligned} I_0 &= x_0^{-4}\int_0^{x_0} x^3 J_0(x)\,\mathrm{d}x + \int_{x_0}^\infty x^{-1}J_0(x)\,\mathrm{d}x, \\ I_1 &= x_0^{-2}\int_0^{x_0} x^2 J_1(x)\,\mathrm{d}x + \int_{x_0}^\infty J_1(x)\,\mathrm{d}x, \end{aligned} \tag{4.15}$$

and

$$x_0 = 2kr_0\cos(\theta/2). \tag{4.16}$$

If we assume, as before, that $kr_0 \ll 1$ then (4.15) become

$$I_0 = \int_{x_0}^{\infty} x^{-1} J_0(x)\,\mathrm{d}x + \tfrac{1}{4} + \mathrm{O}(x_0^2),$$
$$I_1 = \int_0^{\infty} J_1(x)\,\mathrm{d}x - \tfrac{1}{8}x_0^2 + \mathrm{O}(x_0^4). \tag{4.17}$$

The integral in the second of (4.17) is unity and that in the first diverges logarithmically as $x_0 \to 0$, so for sufficiently small kr_0 (4.14) yields

$$f_{\mathrm{Born}}(\theta) = \mathrm{i}\alpha(\pi/2k)^{1/2}[\exp(\mathrm{i}\pi/4)]\{\tan(\theta/2) + \mathrm{O}[\alpha \ln(2kr_0 \cos\tfrac{1}{2}\theta)]\}. \tag{4.18}$$

The term in (4.18) which is linear in α agrees with the calculation of CR. These authors do not consider the term quadratic in α, which diverges in the AB limit, $kr_0 \to 0$. They point out that the linear term in (4.18) differs from the corresponding (small $|\alpha|$) scattering amplitude of AB by the omission of the s-wave contribution to the latter—as we expect from the discussion following (4.8). It is interesting to note that provided θ is not too close to zero (back scattering) the linear term in (4.18) dominates when (4.4) applies, so that in this regime both the small $|\alpha|$ expansion of (4.2), and also that of the total scattering amplitude, are justified. We further note that in this regime the linear term continues to dominate the scattering amplitude in the forward scattering region ($\theta \approx \pi$) where $|f| \to \infty$. This latter divergence is a result of the 'long range' r^{-1} dependence of the vector potential and is not an artefact of either the AB or Born approximations.

Finally, we calculate the phase shift directly using the first Born approximation

$$\delta_n^{\mathrm{Born}} = -\frac{m\pi}{\hbar^2}\int_0^{\infty} J_n^2(kr)\Delta H_n r\,\mathrm{d}r. \tag{4.19}$$

where $\Delta H_n = \Delta H_n(r) = \exp(-\mathrm{i}n\theta)\Delta H \exp(\mathrm{i}n\theta)$. This leads to

$$\delta_n^{\mathrm{Born}} = -\frac{\pi}{2}\left(\int_0^{kr_0} \frac{2\alpha n + \alpha^2 x^2}{k^2 r_0^2} J_n^2(x) x\,\mathrm{d}x + \int_{kr_0}^{\infty} J_n^2(x)\frac{\mathrm{d}x}{x}\right) \tag{4.20}$$

from which we obtain, with $|\alpha|$, $kr_0 \ll 1$,

$$\delta_{n\neq 0}^{\mathrm{Born}} = -\tfrac{1}{2}\pi\alpha n/|n| + \mathrm{O}(\alpha^2, \alpha kr_0, k^2 r_0^2) \tag{4.21}$$

and

$$\delta_0^{\mathrm{Born}} = -\tfrac{1}{2}\pi\alpha^2 \int_{kr_0}^{\infty} J_0^2(x)\frac{\mathrm{d}x}{x} + \mathrm{O}(k^2 r_0^2). \tag{4.22}$$

We observe that (4.22) is consistent with (4.8) and (2.15). The phaseshifts (4.21) agree with the AB result which is obtained by setting $\Delta_n = 0$ in (2.15), verifying the CR observation that the AB and Born scattering amplitudes differ only in their s-wave components.

5. Summary and discussion

We have considered the behaviour of the exact solution (2.9) of the problem of scattering of a beam of electrons by a magnetic solenoid of non-zero radius r_0. Two distinct domains of the wavevector k and flux parameter α were identified, within which approximate solutions based either on the Born approximation or on the $kr_0 \to 0$ solution due to Aharonov and Bohm (1959), are respectively valid. These are, from (3.13) and (4.4) respectively,

$$0 < (kr_0/2)^{2|\alpha|} \ll 1 \qquad \text{AB regime} \tag{5.1}$$

and

$$0 < |\alpha| \ln(2/kr_0) \ll 1 \qquad \text{Born regime} \tag{5.2}$$

where it was assumed $kr_0, |\alpha| \ll 1$. Noting that $x^{2y} = 0.1$ implies $y \ln(1/x) \approx 1.15$ we see that (5.1) is more or less equivalent to

$$|\alpha| \ln(2/kr_0) > 1 \qquad \text{AB regime} \tag{5.3}$$

and the mutually exclusive nature of the two regimes is apparent. To satisfy the AB condition (5.1) or (5.3) requires extremely small values of kr_0 and this regime accounts for only a small fraction of $\alpha - kr_0$ space. Conversely the Born regime applies almost everywhere (in $|\alpha|$, $kr_0 \ll 1$). As a result of their having *no common domain of validity* the AB and Born approximations are seen to be *incompatible*. The paradoxical result of CR comes about due to the (physically meaningless) superposition of two mutually exclusive approximations.

Discrepancies between the AB and Born solutions of an analogous problem in dislocation theory have been reported by Kawamura (1978a) and Kawamura *et al* (1982). These also can be resolved along the above lines. In the simplest (tight-binding) approximation (Kawamura 1978b) the parameter α is given by

$$\alpha = bk_3/2\pi \tag{5.4}$$

where b is the Burgers vector and k_3 is the projection of the wavevector $\boldsymbol{k}$ of a conduction electron parallel to the straight (screw) dislocation. With r_0 now standing for the core radius of the dislocation and k for $k_\perp$, the projection of $\boldsymbol{k}$ perpendicular to the dislocation, the results (5.1) and (5.2) show that the degree of validity of either the Born or AB scattering amplitudes depends on the position of $\boldsymbol{k}$ on the Fermi surface. Except for regions very close to the poles, (5.2) and the Born solution will apply. The logarithmic dependence on $k_\perp$, as opposed to the linear dependence on $|k_3|$, ensures that the AB regime is relatively small. For example, taking $r_0 = (b/\pi)$, and a Fermi radius typical of copper, indicates that $2|\alpha| \ln(2/r_0k_\perp) > 1$ only over small polar caps which account for less than 1% of the Fermi surface area. We may infer that the Born approximation may therefore be used over almost all the Fermi surface for dislocation scattering problems, the exceptions being for waves incident almost parallel to the dislocations.

The thrust of the present paper has been confined to resolving the paradox pointed out by CR. Its contribution to the long and continuing debate on the interpretation of the AB effect (see Erlichson 1970, for a discussion of the earlier debate; Ruijsenaars (1983) provides an extensive bibliography which covers the recent resurgence of interest in this problem) is limited to the following points. Firstly, by resolving the above-mentioned paradox removes some of the aura of mystery surrounding the effect and its interpretation. In particular, by emphasising the pre-emininence of the relatively familiar Born solution we hope to have made more palatable the interpretation of the AB effect as a normal (quantum) scattering process. Secondly, for those who find the interpretation of the AB effect as a scattering process unpalatable, it may help to consider the dislocation analogies referred to above. In these examples, which are mathematically identical to the AB magnetic solenoid problem, the 'reality' of the scattering, now due to the elastic strain field, seems easier to accept. Finally, by showing how one can be misled by a naive application of mathematical techniques to the pseudo-physics encapsulated in the limiting-case differential equation, we question the conclusions of those authors (notably Purcell and Henneberger 1978, Henneberger 1980, 1981, 1984, Henneberger and Huguenin 1981) who base their arguments on a study of the singular case considered by AB.

As in the treatments of AB and CR we have concentrated on the case where no electrostatic potential field is applied. However, the most controversial aspect of this problem concerns the prediction of effects due to the magnetic flux even when the electrons are completely excluded from the flux-containing region by a large potential barrier. It is therefore appropriate to conclude by considering the case (Kretzschmar 1965) where the core potential

$$\begin{aligned} V(r) &= V = \text{constant}, \qquad && r \leqslant r_0 \\ &= 0, && r > r_0 \end{aligned} \tag{5.5}$$

is superposed on the magnetic potential (2.2). This changes the parameter κ of (2.6) to $\kappa_v = [r_0^2 k^2(1-v)/4\alpha] - n/2$, where $v = 2mV/\hbar^2 k^2$. If we retain $|\alpha|$, $r_0 k \ll 1$ and restrict v so that $|r_0^2 k^2(1-v)| \ll 1$ the expression (3.15) is regained for A_n except that on its right-hand side k^2 is replaced by $k^2(1-v)$. As before the $n=0$ term proves to be the most crucial and one finds in place of (3.18)

$$-\frac{b_0}{a_0} = \pi|\alpha|\left[1 + \left(\frac{kr_0}{2}\right)^{-2|\alpha|} \frac{2|\alpha| + vk^2 r_0^2}{2|\alpha| - vk^2 r_0^2}\right]^{-1}, \qquad (|\alpha|,\ kr_0,\ k^2 r_0^2(1-v) \ll 1). \tag{5.6}$$

Thus (3.13) again ensures $|b_0/a_0| \ll \pi|\alpha|$ and the solution in this AB regime is different from that in the Born regime (4.4). Further, according to (5.6), provided $v > 0$ (repulsion) there is another region†, centred on $kr_0 = (2|\alpha|/v)^{1/2}$ and of similar width, within which $|b_0/a_0| \ll \pi|\alpha|$. The AB solution is therefore applicable in this (relatively broad) region but it *turns out to be identical* to that given by the first Born approximation which is readily shown to yield

$$\delta_0^{\text{Born}} = -\frac{\pi}{2}[\tfrac{1}{2}vk^2 r_0^2 + \mathrm{O}(\alpha^2)] \times [1 + \mathrm{O}(k^2 r_0^2)] \tag{5.7}$$

which agrees with (3.1) when $kr_0 = (2|\alpha|/v)^{1/2}$ and $|\alpha|$, $kr_0 \ll 1$. We therefore once again find that the Born solution is adequate outside the very small region (3.13).

Finally, the case $V \to \infty$ is interesting as it models the experimentally important and controversial case of an impenetrable solenoid or whisker. For $k^2 r_0^2(v-1) \gg 1$ we use Abramowitz and Stegun (1970, equation (13.5.14)) and, when $\alpha > 0$, Kummer's transformation (Abramowitz and Stegun 1970, equation (13.1.27)) to find $A_n \sim v^{1/2}$ which yields in (2.9)

$$-\frac{b_n}{a_n} = \frac{J_{|n+\alpha|}(kr_0)}{Y_{|n+\alpha|}(kr_0)}[1 + \mathrm{O}(1/kr_0 v^{1/2})], \qquad kr_0 v^{1/2} \gg 1. \tag{5.8}$$

In the hard-cylinder limit $v \to \infty$ this agrees with Kretzchmar's (1965) equation (41). As in § 3 one readily shows from (5.8) that when kr_0, $|\alpha| \ll 1$

$$-b_0/a_0 = \pi|\alpha|(kr_0/2)^{2|\alpha|}[1 + \mathrm{O}(1/kr_0 v^{1/2})]. \tag{5.9}$$

† The restriction $kr_0 \ll 1$ implies $v \gg |\alpha|$ so this region cannot be followed down to $v = 0$.

For this to agree with the AB solution the condition (5.1) must hold, the *same* condition which was found for $v = 0$! Thus the AB solution is again shown to have a very restricted domain of applicability†. This result may seem surprising for it is tempting to argue (correctly) that with $V \to \infty$ we have $\psi \to 0$ inside $r = r_0$ and therefore the AB boundary condition $\psi(0) = 0$ holds, independent of α and kr_0. However, one cannot conclude from this that the phaseshifts take the AB values (3.1). On the contrary, the phaseshifts are determined by the conditions of continuity *at the boundary* ($r = r_0$), which lead to (2.9). In general, the single condition $\psi(0) = 0$, although true, no more serves to determine the phaseshifts than it does in the familiar hard-cylinder acoustic scattering problem where the phaseshifts certainly depend on kr_0. The exception is when $kr_0 \to 0$, which is expressed by (5.1).

Appendix

The dependence of the coefficient A_n of (2.10) upon kr_0 and α is required for the analysis of the expression (2.9). Using (2.7) we first rewrite (2.10) as

$$kr_0 A_n = |n| - \alpha + (2\alpha\Phi^{-1}\,\partial\Phi/\partial z)_{z=\alpha} \tag{A1}$$

where Φ is given by (2.8) with $c = (1+|n|)$ and $2a = [1+n+|n| - (r_0^2k^2/2\alpha)]$. On using this value of a in the series (2.8), setting $z = \alpha$ and collecting terms of the same order in α we find, if $|\alpha| \ll 1$ and $\gamma \equiv (r_0k/2) \ll 1$,

$$\Phi_{z=\alpha} = \left(1 - \frac{\gamma^2}{c} + O(\gamma^4)\right) + \alpha\left(\frac{\beta}{c} - \frac{\gamma^2}{2}\,\frac{2\beta+1}{c(c+1)} + O(\gamma^4)\right) + O(\alpha^2), \tag{A2}$$

where $2\beta = (1+n+|n|)$. Term by term differentiation of (2.8) leads in a similar way to

$$\left(\alpha\frac{\partial\Phi}{\partial z}\right)_{z=\alpha} = -\frac{\gamma^2}{c}\left(1 - \frac{\gamma^2}{c+1} + O(\gamma^4)\right) + \alpha\left(\frac{\beta}{c} - \frac{\gamma^2(2\beta+1)}{c(c+1)} + O(\gamma^4)\right) + O(\alpha^2), \tag{A3}$$

and use of (A3) and (A2) in (A1) then leads directly to (3.15).

Although (3.15) would suffice to analyse (b_0/a_0) in the Born limit it is preferable to prove (4.7), which does not require $kr_0 \ll 1$. The proof proceeds as above, but with $c = 1$ the power series in γ which were terminated at order γ^2 in (A2) and (A3) are readily summed to all orders in terms of Bessel functions. Then (A1) yields

$$A_0 = [J_0'(kr_0)/J_0(kr_0) + O(\alpha^2)], \tag{A4}$$

which is equivalent to (4.7).

Acknowledgments

Some of the work for the present paper was carried out in the Physics Department of the Universiti Pertanian Malaysia, Serdang, Selangor, Malaysia. The author wishes to thank the head of that Department, Dr Abdul Halim bin Shaari, and his colleagues, for their generous hospitality. Financial support was received from the Australian Universities International Development Programme.

† Of course the (first) Born approximation has no validity at all for this very large perturbation.

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